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One-dimensional Schrödinger operators with $\mathcal{P}$-symmetric zero-range potentials

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Abstract
Non-Hermitian Hamiltonians appearing as operator realizations of $\mathcal{P}$-symmetric ($\mathcal{PT}$-symmetric in physical literature) zero-range singular perturbations of one-dimensional Schrödinger operators are studied. In particular, Hamiltonians with a real spectrum are described in terms of parameters of singular perturbations and, moreover, it is shown that only part of them are similar to Hermitian ones. In this case, they can be used as exactly solvable models of $\mathcal{PT}$-symmetric quantum mechanics.

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1. Introduction

Let $A_0 = -\mathrm{d}^2/\mathrm{d}x^2$ be the second derivative operator with the domain $\mathcal{D}(A_0) = W_2^2(\mathbb{R})$ acting in the space $L_2(\mathbb{R})$.

A one-dimensional Schrödinger operator corresponding to a general zero-range potential at the point $x = 0$ can be given by the expression

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a\langle\delta, \cdot\rangle\delta(x) + b\langle\delta', \cdot\rangle\delta(x) + c\langle\delta, \cdot\rangle\delta'(x) + d\langle\delta', \cdot\rangle\delta'(x), \tag{1}$$

where δ and δ' are, respectively, the Dirac δ-function and its derivative (with support at 0) and a, b, c, d are complex numbers (see, e.g. [1, 2]).

The aim of this paper is to study exactly solvable Hamiltonians corresponding to (1) for the case where the singular potential

$$V = a\langle\delta, \cdot\rangle\delta + b\langle\delta', \cdot\rangle\delta + c\langle\delta, \cdot\rangle\delta' + d\langle\delta', \cdot\rangle\delta',$$

in (1) is not symmetric in the standard sense but satisfies the condition of $\mathcal{P}$-symmetry

$$\mathcal{P}V^* = V\mathcal{P}, \tag{2}$$

where the adjoint V^* is determined by the relation $\langle Vu, v\rangle = \langle u, V^*v\rangle$ $\big(u, v \in W_2^2(\mathbb{R})\big)$ and $\mathcal{P}$ is the space parity operator $\mathcal{P}f(x) = f(-x)$ in $L_2(\mathbb{R})$.

Assuming formally that $\mathcal{T}V = V^*\mathcal{T}$, where $\mathcal{T}$ is the complex conjugation operator $\mathcal{T}f(x) = \overline{f(x)}$, we can reformulate (2) as $\mathcal{PT}V = V\mathcal{PT}$ and, hence, expression (1) is $\mathcal{PT}$-symmetric. From this point of view, the corresponding non-Hermitian operator realizations of (1) can be considered as $\mathcal{PT}$-symmetric Hamiltonians in the framework of the intensively developing $\mathcal{PT}$-symmetric quantum mechanics (see, e.g., [6–8]). However, for such Hamiltonians, we prefer to use the notation $\mathcal{P}$-Hermitian, which is more convenient from the mathematical point of view.

In section 2, using the Albeverio–Kurasov approach [2–4], we obtain a simple description of operator realizations of (1) in terms of parameters a, b, c, d of the singular potential V. We remark that, in contrast to the case of symmetric potentials considered in [5], the obtained Hamiltonians A are $\mathcal{P}$-Hermitian operators in $L_2(\mathbb{R})$, i.e.,

$$\mathcal{P}A^* = A\mathcal{P}, \tag{3}$$

where A^* is the adjoint of A. Such operators are point perturbations of the free Schrödinger operator $A_0 = -\mathrm{d}^2/\mathrm{d}x^2$ and some of them have real spectrum (like Hermitian operators). In this case, they can be considered as $\mathcal{P}$-Hermitian ($\mathcal{PT}$-symmetric) exactly solvable models of $\mathcal{PT}$-symmetric quantum mechanics.

In section 3, we present necessary and sufficient conditions for the reality of the spectra of $\mathcal{P}$-Hermitian realizations of (1) in terms of parameters of the corresponding singular potentials V. These results supplement the spectral analysis of such operators that had been carried out in [9] without answering the question about which $\mathcal{P}$-Hermitian operator corresponds to the fixed potential V in (1).

Let A be a $\mathcal{P}$-Hermitian operator realization of (1) with real spectrum. Since the notion of $\mathcal{P}$-Hermiticity of A is equivalent to the Hermiticity of A with respect to the indefinite metric

$$[f, g] = (\mathcal{P}f, g) = \int_{-\infty}^{\infty} f(-x)\overline{g(x)}\,\mathrm{d}x \qquad (\forall f(x), g(x) \in L_2(\mathbb{R})) \tag{4}$$

one can attempt to develop a consistent quantum theory for such $\mathcal{P}$-Hermitian Hamiltonians. However, in this case, we encounter the difficulty of dealing with a Hilbert space $L_2(\mathbb{R})$ equipped with the indefinite metric (4). Because the norm of states carries a probabilistic interpretation in the standard quantum theory, the presence of an indefinite metric immediately raises problems of interpretation.

For an important class of pseudo-Hermitian Hamiltonians with unbroken spacetime reflection symmetry ($\mathcal{PT}$-symmetry) Bender, Brody and Jones [7] overcome the problem of indefinite metric by the construction of a certain previously unnoted symmetry $\mathcal{C}$ inherent to all pseudo-Hermitian Hamiltonians of such a type.

In analogy with [7], we will say that a $\mathcal{P}$-Hermitian operator A acting in $L_2(\mathbb{R})$ possesses the property of $\mathcal{C}$-symmetry if there exists a bounded linear operator $\mathcal{C}$ in $L_2(\mathbb{R})$ such that the following conditions are satisfied:

(a) $A\mathcal{C} = \mathcal{C}A$;
(b) $\mathcal{C}^2 = I$;
(c) the sesquilinear form $(f, g)_{\mathcal{C}} \equiv [\mathcal{C}f, g]$ $(\forall f, g \in L_2(\mathbb{R}))$ determines an inner product in $L_2(\mathbb{R})$ that is equivalent to the initial one.

The existence of a $\mathcal{C}$-symmetry for a $\mathcal{P}$-Hermitian operator A ensures unitarity of the dynamics generated by A in the norm $\|\cdot\|_{\mathcal{C}}^2 = (\cdot,\cdot)_{\mathcal{C}}$ and it is equivalent to the fact that A is similar to a Hermitian operator (see proposition 1).

Thus, in order to obtain the description of all $\mathcal{P}$-Hermitian Hamiltonians A generated by (1) with previously unnoticed symmetries $\mathcal{C}$ it is sufficient to describe the set of all A that are similar to Hermitian operators.

In section 4, we present simple necessary and sufficient conditions on the parameters of the singular potential V under which the corresponding $\mathcal{P}$-Hermitian operator A is similar to a Hermitian one. It should be noted that some of such operators possess generalized complex eigenvalues, a property that is impossible for the standard Hermitian realizations of (1) with symmetric potentials V. Thus, among the $\mathcal{P}$-Hermitian operator realizations of (1) there exist operators similar to Hermitian one but their spectral properties cannot be described in terms of Hermitian realizations of (1).

In section 5, we find the explicit form of $\mathcal{C}$-symmetries for some $\mathcal{P}$-Hermitian Hamiltonians generated by (1) in terms of parameters of the singular potential V.

We remark that a previously unnoted symmetry $\mathcal{C}$ depends on the choice of A and finding $\mathcal{C}$ in explicit form for various classes of pseudo-Hermitian operators is a non-trivial problem that, as a rule, requires additional assumptions on the structure of spectrum (see e.g. [8], where the case of diagonalizable $\mathcal{P}$-Hermitian Hamiltonians with discrete spectrum was considered). However, the spectral restrictions can be omitted if we consider $\mathcal{P}$-Hermitian Hamiltonians generated by (1).

Let us make a remark about notation. $\mathcal{D}(A)$ and $A\upharpoonright_{\mathcal{D}}$ denote the domain of a linear operator A and the restriction of A onto a set $\mathcal{D}$, respectively. The symbol $W_2^p(\mathbb{R})$ ($p \in \{-2, 2\}$) denotes the usual Sobolev space, i.e. $W_2^p(\mathbb{R})$ is the space of tempered distributions with a Fourier transform which is square integrable with respect to the measure with density $(1+|x|^2)^{p/2}$.

2. $\mathcal{P}$-Hermitian operator realizations

It is clear that the heuristic expression (1) determines the symmetric operator

$$A_{\mathrm{sym}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad \mathcal{D}(A_{\mathrm{sym}}) = \left\{u(x) \in W_2^2(\mathbb{R}) \,\middle|\, u(0) = u'(0) = 0\right\} \tag{5}$$

in $L_2(\mathbb{R})$. Thus, any proper extension A of A_{sym} (i.e., $A_{\mathrm{sym}} \subset A \subset A_{\mathrm{sym}}^*$) can be considered as an operator realization of (1) in $L_2(\mathbb{R})$.

In order to specify more exactly which a proper extension A of A_{sym} corresponds to (1) with a given $\mathcal{P}$-symmetric singular potential V, we will use an approach suggested in [3, 4]. The main idea consists in the construction of some regularization $\mathbb{A}_{\mathbf{R}}$ of (1) that is well defined as an operator from $\mathcal{D}(A_{\mathrm{sym}}^*) = W_2^2(\mathbb{R}\backslash\{0\})$ to $W_2^{-2}(\mathbb{R})$. Then, the corresponding operator realization A of (1) is determined as follows:

$$A = \mathbb{A}_{\mathbf{R}}\upharpoonright_{\mathcal{D}(A)}, \qquad \mathcal{D}(A) = \{f \in \mathcal{D}(A_{\mathrm{sym}}^*) \mid \mathbb{A}_{\mathbf{R}} f \in L_2(\mathbb{R})\}. \tag{6}$$

To obtain a regularization of (1) it suffices to extend the distributions δ and δ' onto $W_2^2(\mathbb{R}\backslash\{0\})$. The most reasonable way (based on preserving of initial homogeneity of δ and δ' with respect to scaling transformations, see, for details, [2, 12]) leads to the following definition:

$$\langle \delta_{\mathrm{ex}}, f\rangle = \frac{f(+0)+f(-0)}{2}, \qquad \langle \delta_{\mathrm{ex}}', f\rangle = -\frac{f'(+0)+f'(-0)}{2} \tag{7}$$

for all $f(x) \in W_2^2(\mathbb{R}\backslash\{0\})$. In this case, the regularization of (1) onto $W_2^2(\mathbb{R}\backslash\{0\})$ has the form

$$\mathbb{A}_{\mathbf{R}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + a\langle\delta_{\mathrm{ex}}, \cdot\rangle\delta(x) + b\langle\delta'_{\mathrm{ex}}, \cdot\rangle\delta(x) + c\langle\delta_{\mathrm{ex}}, \cdot\rangle\delta'(x) + d\langle\delta'_{\mathrm{ex}}, \cdot\rangle\delta'(x),$$

where $-\mathrm{d}^2/\mathrm{d}x^2$ acts on $W_2^2(\mathbb{R}\backslash\{0\})$ in the distributional sense.

Extending $\mathcal{P}$ onto $W_2^{-2}(\mathbb{R})$, one gets $\mathcal{P}\delta = \delta$ and $\mathcal{P}\delta' = -\delta'$. These relations and (1) imply that the condition of $\mathcal{P}$-symmetry (2) is equivalent to the following restrictions on the parameters of V:

$$a, d \text{ are real} \qquad \text{and} \qquad c = -\overline{b}.$$

In what follows, we assume that V satisfies such conditions.

Theorem 1. *The operator realization of (1) with $\mathcal{P}$-symmetric potential V is a $\mathcal{P}$-Hermitian operator A that coincides with the restriction of $A^*_{\mathrm{sym}} = -\mathrm{d}^2/\mathrm{d}x^2$ onto the domain*

$$\mathcal{D}(A) = \left\{f(x) \in W_2^2(\mathbb{R}\backslash\{0\}) \,\middle|\, \mathbf{T}\Gamma_0 f = \Gamma_1 f\right\}, \qquad \mathbf{T} = \begin{pmatrix} a & b \\ -\overline{b} & d \end{pmatrix}, \tag{8}$$

where

$$\Gamma_0 f = \frac{1}{2}\begin{pmatrix} f(+0) + f(-0) \\ -f'(+0) - f'(-0) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f'(+0) - f'(-0) \\ f(+0) - f(-0) \end{pmatrix}. \tag{9}$$

Proof. Let us consider the functions

$$h_1(x) = \begin{cases} \mathrm{e}^{-x}, & x > 0 \\ \mathrm{e}^{x}, & x < 0 \end{cases} \qquad h_2(x) = \begin{cases} -\mathrm{e}^{-x}, & x > 0 \\ \mathrm{e}^{x}, & x < 0. \end{cases}$$

It is clear that any function $f \in W_2^2(\mathbb{R}\backslash\{0\})$ can be represented as $f(x) = u(x) + \sum_{j=1}^2 \xi_j h_j(x)$, $\left(u \in \overset{0}{W_2^2}(\mathbb{R}), \xi_i \in \mathbb{C}\right)$ and

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}h_1(x) = -h_1(x) + 2\delta(x), \qquad -\frac{\mathrm{d}^2}{\mathrm{d}x^2}h_2(x) = -h_2(x) + 2\delta'(x). \tag{10}$$

Applying $\mathbb{A}_{\mathbf{R}}$ to $f(x)$ and taking into account (7), (9), (10), and the relations

$$f'(+0) - f'(-0) = -2\xi_1, \qquad f(+0) - f(-0) = -2\xi_2,$$

we obtain

$$\begin{aligned}\mathbb{A}_{\mathbf{R}} f(x) &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) - \sum_{j=1}^{2}\xi_j h_j(x) + 2\xi_1\delta(x) + 2\xi_2\delta'(x) + \frac{1}{2}(f(+0) \\ &\quad + f(-0))(a\delta(x) - \overline{b}\delta'(x)) - \frac{1}{2}(f'(+0) + f'(-0))(b\delta(x) + d\delta'(x)) \\ &= A^*_{\mathrm{sym}} f(x) + (\delta(x), \delta'(x))(\mathbf{T}\Gamma_0 f - \Gamma_1 f).\end{aligned}$$

This equality and (6) imply that the operator realization A of (1) is defined by (8).

Let us show that A satisfies the condition of $\mathcal{P}$-Hermiticity (3). We start from the observation that $\mathcal{P}$ commutates with A^*_{sym} and, hence, condition (3) is equivalent to the relation $\mathcal{P}\mathcal{D}(A^*) = \mathcal{D}(A)$.

Using (8), it is easy to verify that

$$\mathcal{D}(A^*) = \left\{f \in W_2^2(\mathbb{R}\backslash\{0\}) \,\middle|\, \overline{\mathbf{T}}^{\mathrm{t}}\Gamma_0 f = \Gamma_1 f\right\}, \qquad \overline{\mathbf{T}}^{\mathrm{t}} = \begin{pmatrix} a & -b \\ \overline{b} & d \end{pmatrix}. \tag{11}$$

The validity of $\mathcal{P}\mathcal{D}(A^*) = \mathcal{D}(A)$ immediately follows from (8) and (11) if we take into account that $\mathbf{G}\mathbf{T} = \overline{\mathbf{T}}^t\mathbf{G}$ and

$$\Gamma_0\mathcal{P}f = \mathbf{G}\Gamma_0 f, \qquad \Gamma_1\mathcal{P}f = \mathbf{G}\Gamma_1 f, \qquad \forall f \in \mathcal{D}(A^*_{\mathrm{sym}}),$$

where $\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus A is a $\mathcal{P}$-Hermitian operator. Theorem 1 is proved. □

Remarks.

1. A similar approach to the definition of Hermitian operator realizations in terms of mean values $\Gamma_0 f$ and jumps $\Gamma_1 f$ of functions $f(x) \in W_2^2(\mathbb{R}\backslash\{0\})$ has recently been suggested by Albeverio and Nizhnik [13].
2. Another description of $\mathcal{P}$-Hermitian extensions of A_{sym} ($\mathcal{PT}$-self-adjoint point perturbations) was obtained in [9] with the use of boundary operators

$$\Gamma_0 f = \begin{pmatrix} f(-0) \\ f'(-0) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f(+0) \\ f'(+0) \end{pmatrix}.$$

3. Spectral analysis

It was shown in [9] that the continuous spectrum of any $\mathcal{P}$-Hermitian extension A of A_{sym} coincides with $[0, \infty)$ and only the point spectrum of A can be situated in $\mathbb{C}\backslash\mathbb{R}_+$.

Using theorem 1, we can supplement the results of [9] and to obtain a description of non-real spectra of $\mathcal{P}$-Hermitian operator realizations of (1) in terms of the parameters a, b, d of the singular potential V.

Theorem 2. *The $\mathcal{P}$-Hermitian operator A defined by (8) has points of non-real spectrum if and only if one of the following conditions is satisfied:*

(i) $D \equiv ((|b| + 2)^2 + ad)((|b| - 2)^2 + ad) < 0$, $(4 - |b|^2 - ad)d > 0$;
(ii) $a = d = 0$, $|b| = 2$.

Condition (i) corresponds to the case where A has two non-real eigenvalues, which are conjugate to each other. Condition (ii) describes the situation where any point $z \in \mathbb{C}\backslash\mathbb{R}_+$ is an eigenvalue of A.

Proof. Let us denote by τ the square root of the energy parameter $z = \tau^2$ determined uniquely by the condition $\mathrm{Im}\,\tau > 0$ and consider the functions

$$h_{1\tau}(x) = \begin{cases} \mathrm{e}^{\mathrm{i}\tau x}, & x > 0 \\ \mathrm{e}^{-\mathrm{i}\tau x}, & x < 0 \end{cases} \qquad h_{2\tau}(x) = \begin{cases} -\mathrm{e}^{\mathrm{i}\tau x}, & x > 0 \\ \mathrm{e}^{-\mathrm{i}\tau x}, & x < 0 \end{cases} \tag{12}$$

that form a basis of $\ker(A^*_{\mathrm{sym}} - zI)$, where $z = \tau^2$ runs $\mathbb{C}\backslash\mathbb{R}_+$. It is clear that z belongs to the point spectrum of A if and only if there exists a function $f \in \ker(A^*_{\mathrm{sym}} - zI) \cap \mathcal{D}(A)$. Representing $f(x)$ in the form $f(x) = c_1 h_{1\tau}(x) + c_2 h_{2\tau}(x)$ ($c_i \in \mathbb{C}$) and substituting this expression into (8) we arrive at the conclusion that z is an eigenvalue of A if and only if the system of equations

$$(a - 2\mathrm{i}\tau)c_1 + \mathrm{i}b\tau c_2 = 0 \qquad \overline{b}c_1 - (\mathrm{i}d\tau + 2)c_2 = 0$$

has a nontrivial solution c_1, c_2. This is possible if the determinant of the coefficient matrix of the system is equal to zero, i.e.,

$$2d\tau^2 + \mathrm{i}(ad + |b|^2 - 4)\tau + 2a = 0. \tag{13}$$

For $d \neq 0$, the roots of (13) are

$$\tau_{1,2} = \mathrm{i}\frac{4 - |b|^2 - ad \pm \sqrt{D}}{4d}, \tag{14}$$

where D is determined above. Thus, for $d \neq 0$, condition (i) is necessary and sufficient for the existence of two non-real eigenvalues $z_{1,2} = \tau_{1,2}^2$ of A, which are conjugate to each other.

Similarly, for $d = 0$, condition (ii) is necessary and sufficient for the existence of non-real eigenvalues. In this case, the left-hand side of (13) vanishes and, for any $\tau(\operatorname{Im}\tau > 0)$, a nontrivial solution of the system above can be chosen as follows $c_1 = b, c_2 = |b|$. Thus, any point $z = \tau^2 \in \mathbb{C}\backslash\mathbb{R}_+$ is an eigenvalue of A. The corresponding eigenfunction $f_z(x) = bh_{1\tau}(x) + |b|h_{2\tau}(x)$ takes the form

$$f_z(x) = \begin{cases} (b - |b|)\,\mathrm{e}^{\mathrm{i}\tau x}, & x > 0 \\ (b + |b|)\,\mathrm{e}^{-\mathrm{i}\tau x}, & x < 0. \end{cases}$$

Theorem 2 is proved. □

Let us consider a $\mathcal{P}$-Hermitian operator A defined by (8) where the parameters a, b, d satisfy the inequalities

$$D < 0, \qquad (4 - |b|^2 - ad)d < 0. \tag{15}$$

We remark that the condition $D < 0$ ensures $d \neq 0$. Hence, (14) determines the roots $\tau_{1,2}$ of equation (13). In this case, $\tau_{1,2}$ lie on the nonphysical sheet $\operatorname{Im}\tau \leqslant 0$ and they do not determine eigenvalues $z_{1,2} = \tau_{1,2}^2$ of A because the corresponding eigenfunctions

$$f_{z_j}(x) = \begin{cases} \left(1 - \frac{\overline{b}}{\mathrm{i}d\tau_j+2}\right)\mathrm{e}^{\mathrm{i}\tau_j x}, & x > 0 \\ \left(1 + \frac{\overline{b}}{\mathrm{i}d\tau_j+2}\right)\mathrm{e}^{-\mathrm{i}\tau_j x}, & x < 0 \end{cases} \qquad j = 1, 2$$

do not belong to $L_2(\mathbb{R})$.

Thus, any $\mathcal{P}$-Hermitian operator A defined by (8) with additional condition (15) on the parameters a, b, d has continuous spectrum on $[0, \infty)$ and the pair of generalized complex eigenvalues $z_{1,2}$ in the sense that the corresponding eigenfunctions $f_{z_{1,2}}(x)$ satisfy at the origin the boundary conditions that determine A but $f_{z_{1,2}} \notin L_2(\mathbb{R})$.

In the next section, we show that any $\mathcal{P}$-Hermitian operator A of such a type is similar to a Hermitian one. At the same time, it is impossible to construct standard Hermitian realizations of (1) having generalized complex eigenvalues. Let us explain this fact in more details. Indeed, if V is a symmetric singular potential (i.e., $V^* = V$), then its parameters a, b, c, d satisfy the conditions $a, d \in \mathbb{R}$, $c = \overline{b}$ and the corresponding Hermitian operator realization A of (1) is defined by (8), where $\mathbf{T} = \left(\begin{smallmatrix} a & b \\ \overline{b} & d \end{smallmatrix}\right)$. For such a Hermitian operator A, repeating the proof of theorem 2, we conclude that $z = \tau^2$ is an eigenvalue of A if and only if $2d\tau^2 + \mathrm{i}(ad - |b|^2 - 4)\tau + 2a = 0$. Since the roots $r_{1,2}$ of this equation are purely imaginary for any choice of a, b, d, we get that A cannot have generalized complex eigenvalues $z_{1,2} = \tau_{1,2}^2$.

Thus, the study of $\mathcal{P}$-Hermitian operator realizations of (1) allows one to obtain exactly solvable Hamiltonians with properties that cannot be described with the use of standard Hermitian operator realizations of (1).

4. Conditions of similarity

4.1. Auxiliary statements

We recall that an operator A is similar to a Hermitian operator H if there exists an invertible bounded operator Z such that $H = ZAZ^{-1}$. If Z is a uniformly positive[6] Hermitian operator, then, setting $F = Z^2$, it is easy to verify the following statement:

Lemma 1. *If there exists a bounded uniformly positive Hermitian operator F such that $A^*F = FA$, then A is similar to the Hermitian operator $H = \sqrt{F}A\sqrt{F}^{-1}$.*

At the present time, there are various approaches to the study of the problem of similarity. One of them is based on a general integral-resolvent criterion of similarity obtained independently in [10, 11]. This criterion is especially useful when A is a finite-dimensional perturbation of a Hermitian operator (see, e.g., [11, 14]) and for the case of $\mathcal{P}$-Hermitian operators it can be written as follows:

Theorem 3. *A $\mathcal{P}$-Hermitian operator A acting in $L_2(\mathbb{R})$ is similar to a Hermitian one if and only if the spectrum of A is real and there exists a constant M such that*

$$\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(A - zI)^{-1} g(x)\|^2 \, \mathrm{d}\xi \leqslant M\|g(x)\|^2, \qquad z = \xi + \mathrm{i}\varepsilon, \quad \forall g \in L_2(\mathbb{R}),$$

where the integral is taken along the line $z = \xi + \mathrm{i}\varepsilon$ ($\varepsilon > 0$ is fixed).

In order to apply theorem 3, we need an explicit form of the resolvent $(A - zI)^{-1}$.

Lemma 2. *Let A be a $\mathcal{P}$-Hermitian operator defined by (8) and let $A_0 = -\mathrm{d}^2/\mathrm{d}x^2$, $\mathcal{D}(A_0) = W_2^2(\mathbb{R})$ be the free Schrödinger operator. Then, for all $g_\pm \in L_2(\mathbb{R}_\pm)$ and for all $z = \tau^2$ from the resolvent set $\rho(A)$ of A,*

$$(A - zI)^{-1} g_\pm(x) = (A_0 - zI)^{-1} g_\pm(x) + c_{1\pm}(\tau)h_{1\tau}(x) + c_{2\pm}(\tau)h_{2\tau}(x), \quad x \in \mathbb{R},$$

where $h_{j\tau}(x)$ are defined by (12) and

$$c_{1\pm}(\tau) = \frac{\mathrm{i}F_\pm(\tau)}{\tau}\left(-1 + \frac{2d\tau^2 - 2\mathrm{i}\tau(2 \pm b)}{p(\tau)}\right),$$

$$c_{2\pm}(\tau) = \pm\frac{\mathrm{i}F_\pm(\tau)}{\tau}\left(-1 + \frac{-2\mathrm{i}\tau(2 \pm \overline{b}) + 2a}{p(\tau)}\right),$$

where $F_\pm(\tau) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{\pm\mathrm{i}\tau s} g_\pm(s)\,\mathrm{d}s$ and $p(\tau) = 2d\tau^2 + \mathrm{i}(ad + |b|^2 - 4)\tau + 2a$.

Proof. Since A and A_0 are proper extensions of A_{sym} and $h_{j\tau}(x)$ ($j = 1, 2$) form a basis of $\ker(A^*_{\mathrm{sym}} - zI)$, we get

$$(A - zI)^{-1} g(x) = (A_0 - zI)^{-1} g(x) + c_1(\tau)h_{1\tau}(x) + c_2(\tau)h_{2\tau}(x), \qquad \forall g \in L_2(\mathbb{R}),$$

where $c_j(\tau)$ are two parameters to be calculated.

Let us rewrite the latter equality as follows:

$$f(x) = f_0(x) + c_1(\tau)h_{1\tau}(x) + c_2(\tau)h_{2\tau}(x), \tag{16}$$

where $f(x) = (A - zI)^{-1} g(x) \in \mathcal{D}(A)$ and $f_0(x) = (A_0 - zI)^{-1} g(x) \in \mathcal{D}(A_0) = W_2^2(\mathbb{R})$.

[6] A Hermitian operator Z is called uniformly positive if there exists $m > 0$ such that $Z \geqslant mI$.

It follows from (9), (12) and (16) that

$$\Gamma_0 f = \begin{pmatrix} f_0(0) \\ -f_0'(0) \end{pmatrix} + \begin{pmatrix} c_1(\tau) \\ \mathrm{i}\tau c_2(\tau) \end{pmatrix}, \qquad \Gamma_1 f = 2\begin{pmatrix} \mathrm{i}\tau c_1(\tau) \\ -c_2(\tau) \end{pmatrix}$$

Substituting the values of $\Gamma_j f$ into (8) and solving the obtained equation with respect to $c_j(\tau)$, we get

$$\begin{pmatrix} c_1(\tau) \\ c_2(\tau) \end{pmatrix} = \frac{-1}{p(\tau)} \begin{pmatrix} \mathrm{i}\tau(|b|^2 + ad) + 2a & 2b \\ 2\mathrm{i}\bar{b}\tau & |b|^2 + ad - 2d\,\mathrm{i}\tau \end{pmatrix} \begin{pmatrix} f_0(0) \\ -f_0'(0) \end{pmatrix}. \tag{17}$$

Recalling the well-known formula

$$f_0(x) = (A_0 - zI)^{-1} g(x) = \frac{\mathrm{i}}{2\tau} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\tau|x-s|} g(s)\,\mathrm{d}s,$$

we obtain

$$\begin{pmatrix} f_0(0) \\ -f_0'(0) \end{pmatrix} = \frac{1}{2} \int_{\mathbb{R}} \mathrm{e}^{\pm\mathrm{i}\tau s} g_\pm(s)\,\mathrm{d}s \begin{pmatrix} \mathrm{i}/\tau \\ \mp 1 \end{pmatrix} = F_\pm(\tau) \begin{pmatrix} \mathrm{i}/\tau \\ \mp 1 \end{pmatrix},$$

where $g = g_- \in L_2(\mathbb{R}_-)$ or $g = g_+ \in L_2(\mathbb{R}_+)$.

Now, to complete the proof of lemma 2, it suffices to substitute the obtained values of $f_0(0)$ and $f_0'(0)$ into (17) and carry out the trivial transformations. □

4.2. Conditions of similarity

Theorem 4. *Let A be a $\mathcal{P}$-Hermitian operator realization of (1) defined by (8) and let the corresponding parameters a, b, d satisfy one of the following conditions:*

(i) $D < 0, \quad (4 - |b|^2 - ad)d = 0$;
(ii) $D = 0, \quad (4 - |b|^2 - ad)d = 0$;
(iii) $D = 0, \quad (4 - |b|^2 - ad)d > 0, \quad b \neq 0,$

where $D = ((|b| + 2)^2 + ad)((|b| - 2)^2 + ad)$. Then the operator A has real spectrum (except the extremal case $a = d = 0, |b| = 2$ (see theorem 2)) but A is not similar to a Hermitian operator.

Proof. If parameters a, b, d satisfy one of the conditions (i)–(iii) of theorem 4, then they cannot satisfy conditions of theorem 2 (except the extremal case $a = d = 0$, $|b| = 2$). Hence, the corresponding operator A defined by (8) has only a real spectrum.

Let us assume that such an operator A is similar to a Hermitian one. Then, for all $g(x) \in L_2(\mathbb{R})$ and $z = \tau^2 \in \mathbb{C}\backslash\mathbb{R}$,

$$\|((A - zI)^{-1} - (A_0 - zI)^{-1})g(x)\|^2 \leqslant \frac{M}{(\operatorname{Im} z)^2}\|g(x)\|^2, \tag{18}$$

where M is a constant independent of $g(x)$ and z. In particular, inequality (18) will be true if we put $g = g_+$ or $g = g_-$, where

$$g_+(x) = \begin{cases} \mathrm{e}^{-\mathrm{i}\bar{\tau}x}, & x > 0 \\ 0, & x < 0, \end{cases} \qquad g_-(x) = \begin{cases} 0, & x > 0 \\ \mathrm{e}^{\mathrm{i}\bar{\tau}x}, & x < 0 \end{cases}$$

In these cases, using lemma 2 and taking into account that

$$\|g_\pm(x)\|^2 = \frac{1}{2(\operatorname{Im}\tau)}, \qquad \|h_{j\tau}(x)\|^2 = \frac{1}{\operatorname{Im}\tau}, \qquad F_\pm(\tau) = \frac{1}{4(\operatorname{Im}\tau)},$$

and $(\operatorname{Im} z)^2 = 4(\operatorname{Im}\tau)^2(\operatorname{Re}\tau)^2$ we can rewrite (18) as follows

$$\Phi_\pm(\tau) \leqslant 2M, \qquad \forall\tau \in \mathcal{L} = \{\operatorname{Im}\tau > 0,\ \operatorname{Re}\tau \neq 0\},$$

where

$$\Phi_\pm(\tau) = \frac{(\operatorname{Re}\tau)^2}{|\tau|^2}\left(\left|1 - \frac{2d\tau^2 - 2\mathrm{i}\tau(2 \pm b)}{p(\tau)}\right|^2 + \left|1 - \frac{-2\mathrm{i}\tau(2 \pm \overline{b}) + 2a}{p(\tau)}\right|^2\right)$$

are continuous functions on $\mathcal{L}$. Thus, the property of A to be similar to a Hermitian operator implies the uniform boundedness of $\Phi_+(\tau)$ and $\Phi_-(\tau)$ for all $\tau \in \mathcal{L}$.

Assume that the parameters a, b, d satisfy condition (i) of theorem 4. Then, by virtue of (14), the roots $\tau_{1,2}$ of $p(\tau)$ are real and have the form $\tau_{1,2} = \pm\sqrt{|D|}/4d$. In this case, the functions $\Phi_+(\tau)$ and $\Phi_-(\tau)$ tend to infinity in a neighbourhood at least one of the points τ_1 and τ_2. Thus, if condition (i) holds, then the operator A cannot be similar to an Hermitian one.

Considering similarly the cases where condition (ii) or (iii) is true, we arrive at the conclusion that at least one of the functions $\Phi_\pm(\tau)$ is not uniformly bounded on $\mathcal{L}$. Thus A is not similar to a Hermitian operator. Theorem 4 is proved. □

Analysing the conditions for parameters a, b, d in theorems 2 and 4, it is easy to see that the other possible relations between a, b, d can be written as follows:

(i) $b = 0$;
(ii) $D > 0, \quad b \neq 0$;
(iii) $D < 0, \quad (4 - |b|^2 - ad)d < 0$;
(iv) $D = 0, \quad (4 - |b|^2 - ad)d < 0, \quad b \neq 0$;

It turns out that conditions (i)–(iv) are *necessary and sufficient* for the similarity of the corresponding $\mathcal{P}$-Hermitian operator A defined by (8) to a Hermitian one. Indeed, in case (i), the matrix $\mathbf{T}$ appearing in theorem 1 is Hermitian and, hence, the operator A defined by (8) is also Hermitian (see [13] for details). Case (ii) will be considered in theorem 6 with the use of lemma 1. In the theorem presented below, we prove the similarity for the cases where A possesses generalized eigenvalues (cases (iii) and (iv)) with the use of the general integral-resolvent criterion of similarity [10, 11].

Theorem 5. *Let A be a $\mathcal{P}$-Hermitian operator realization of (1) defined by (8) and let the corresponding parameters a, b, d satisfy one of condition (iii) or (iv). Then A is similar to a Hermitian operator.*

Proof. Assume that A is a $\mathcal{P}$-Hermitian operator defined by (8), where the corresponding parameters a, b, d satisfy one of the conditions (iii), (iv). Then, by virtue of theorem 2, the spectrum of A is real. Furthermore, by theorem 3, the existence of a constant M such that

$$\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(A - zI)^{-1}g(x) - (A_0 - zI)^{-1}g(x)\|^2\,\mathrm{d}\xi \leqslant M\|g(x)\|^2 \tag{19}$$

(for all $g \in L_2(\mathbb{R})$) implies the similarity of A to a Hermitian operator.

Let $g(x) = g_+(x)$ be an arbitrary function from $L_2(\mathbb{R}_+)$. Using lemma 2 and the relation $\|h_{j\tau}(x)\|^2 = 1/\operatorname{Im}\tau$ (see the proof of theorem 4), we get

$$\|(A - zI)^{-1}g_+(x) - (A_0 - zI)^{-1}g_+(x)\|^2 = \frac{|F_+(\tau)|^2}{|\tau|^2(\operatorname{Im}\tau)} M_+(\tau), \tag{20}$$

where

$$M_+(\tau) = \left|1 - \frac{2d\tau^2 - 2\mathrm{i}\tau(2 + b)}{p(\tau)}\right|^2 + \left|1 - \frac{-2\mathrm{i}\tau(2 + \overline{b}) + 2a}{p(\tau)}\right|^2.$$

Let us analyse the components of the right-hand side of (20). First of all we note that if condition (iii) or (iv) holds, then the roots $\tau_{1,2}$ of $p(\tau)$ belong to the lower half-plane ($\operatorname{Im}\tau < 0$) and, hence, there exists a constant M_1 such that $M_+(\tau) < M_1$ for all τ from the upper half-plane ($\operatorname{Im}\tau > 0$).

Next, by definition (see lemma 2), $F_+(\tau)$ is the Fourier transform of $g_+(x) \in L_2(\mathbb{R}_+)$. Thus, $F_+(\tau)$ belongs to the Hardy space $H^2(\mathbb{C}_+)$.

Finally, we remark that $z = \tau^2$ is changed along the straight line $z = \xi + \mathrm{i}\varepsilon$ ($\xi \in \mathbb{R}, \varepsilon > 0$ is fixed) if and only if $\operatorname{Im}\tau = \varepsilon/2\operatorname{Re}\tau$ and $\xi = (\operatorname{Re}\tau)^2 - \varepsilon^2/4(\operatorname{Re}\tau)^2$, where the variable $\operatorname{Re}\tau$ runs $[0, +\infty)$.

Using equality (20), the remarks above, and taking into account the Carleson embedding theorem [15, section VIII], we get

$$\varepsilon\int_{-\infty}^{\infty}\|(A - zI)^{-1}g_+(x) - (A_0 - zI)^{-1}g_+(x)\|^2\,\mathrm{d}\xi < M_1\int_{-\infty}^{\infty}\frac{\varepsilon|F_+(\tau)|^2}{|\tau|^2(\operatorname{Im}\tau)}\,\mathrm{d}\xi$$
$$= 4M_1\int_0^{\infty}|F_+(\tau)|^2\,\mathrm{d}\operatorname{Re}\tau \leqslant M_+\|F_+\|^2_{H^2(\mathbb{C}_+)} = \frac{\pi}{2}M_+\|g_+(x)\|^2,$$

where M_+ is a constant independent of $\varepsilon > 0$ and $g_+(x)$.

Thus, inequality (19) holds for any $g_+(x) \in L_2(\mathbb{R}_+)$. Considering similarly the case where $g(x) = g_-(x) \in L_2(\mathbb{R}_-)$ and choosing $M = \frac{\pi}{2}\max\{M_-, M_+\}$, we arrive at the conclusion that (19) is true for all functions from $L_2(\mathbb{R})$ and, hence, A is similar to a Hermitian operator. Theorem 5 is proved. □

We will say that a bounded operator K is *an operator of transition* if K is a Hermitian strong contraction (i.e., $K = K^*$, $\|K\| < 1$) and $\mathcal{P}K = -K\mathcal{P}$.

Let us consider a collection of operators $K_{\theta,\omega}$ acting in $L_2(\mathbb{R})$ and defined by the formula

$$K_{\theta,\omega} : (I \pm \mathcal{P})f(x) \to \mathrm{e}^{\pm\mathrm{i}\omega}\frac{1-\theta}{1+\theta}(\operatorname{sign} x)(I \pm \mathcal{P})f(x), \qquad \forall f(x) \in L_2(\mathbb{R}),$$

where $\theta > 0$ and $\omega \in [0, 2\pi)$. It is easy to verify that $K_{\theta,\omega}$ is an operator of transition for any choice of parameters θ and ω and the operator $F_{\theta,\omega} = (I - K_{\theta,\omega})(I + K_{\theta,\omega})^{-1}$ has the form

$$F_{\theta,\omega}f(x) = \alpha f(x) + \frac{\beta}{2}(\operatorname{sign} x)(\mathrm{e}^{\mathrm{i}\omega}(I + \mathcal{P}) + \mathrm{e}^{-\mathrm{i}\omega}(I - \mathcal{P}))f(x), \tag{21}$$

where $\alpha = (\theta + 1/\theta)/2$ and $\beta = (\theta - 1/\theta)/2$.

Theorem 6. *Let A be a $\mathcal{P}$-Hermitian operator realization of (1) defined by (8) and let the corresponding parameters a, b, d satisfy the conditions: $D > 0, b \neq 0$, where $D = ((|b| + 2)^2 + ad)((|b| - 2)^2 + ad)$. Then A is similar to the Hermitian operator $H = F_{\sqrt{\theta},\omega}AF_{\sqrt{\theta},\omega_1}$, where parameters $\theta > 0$, $\omega, \omega_1 \in [0, 2\pi)$ are determined by the relations*

$$\theta = \sqrt{\frac{(|b|+2)^2 + ad}{(|b|-2)^2 + ad}}, \qquad \mathrm{e}^{\mathrm{i}\omega} = \frac{|b|}{b}, \qquad |\omega - \omega_1| = \pi. \tag{22}$$

Proof. The definition of $F_{\theta,\omega}$ in terms of operators of transition $K_{\theta,\omega}$ immediately implies that $F_{\theta,\omega}$ is a bounded uniformly positive Hermitian operator and $(\mathcal{P}F_{\theta,\omega})^2 = I$. But then, by virtue of lemma 1, the equality

$$A^*F_{\theta,\omega} = F_{\theta,\omega}A \tag{23}$$

ensures the similarity of A to the Hermitian operator $H = \sqrt{F_{\theta,\omega}}A\sqrt{F_{\theta,\omega}}^{-1}$.

Relations (5) and (21) yield that $A^*_{\rm sym}F_{\theta,\omega} = F_{\theta,\omega}A^*_{\rm sym}$ for any choice of parameters θ and ω. Hence, equalities (23) and $F_{\theta,\omega}\mathcal{D}(A) = \mathcal{D}(A^*)$ are equivalent. In view of relations $(\mathcal{P}F_{\theta,\omega})^2 = I$, (3), and (11), the latter equality is equivalent to the following statement:

$$\text{if}\quad \mathbf{T}\Gamma_0 f = \Gamma_1 f, \quad \text{then}\quad \overline{\mathbf{T}}^{\rm t}\Gamma_0 F_{\theta,\omega} f = \Gamma_1 F_{\theta,\omega} f. \tag{24}$$

Using (9) and (21), it is easy to verify that

$$\Gamma_0 F_{\theta,\omega} f = \alpha\Gamma_0 f + \frac{\beta}{2}\begin{pmatrix} 0 & {\rm e}^{-{\rm i}\omega} \\ -{\rm e}^{{\rm i}\omega} & 0\end{pmatrix}\Gamma_1 f, \qquad \Gamma_1 F_{\theta,\omega} f = 2\beta\begin{pmatrix} 0 & -{\rm e}^{-{\rm i}\omega} \\ {\rm e}^{{\rm i}\omega} & 0\end{pmatrix}\Gamma_0 f + \alpha\Gamma_1 f$$

for all $f \in W_2^2(\mathbb{R}\backslash\{0\})$. Substituting these expressions into (24), we arrive at the conclusion that (24) is equivalent to the following matrix equality:

$$\alpha(\mathbf{T} - \overline{\mathbf{T}}^{\rm t}) = \frac{\beta}{2}\overline{\mathbf{T}}^{\rm t}\begin{pmatrix} 0 & {\rm e}^{-{\rm i}\omega} \\ -{\rm e}^{{\rm i}\omega} & 0\end{pmatrix}\mathbf{T} - 2\beta\begin{pmatrix} 0 & -{\rm e}^{-{\rm i}\omega} \\ {\rm e}^{{\rm i}\omega} & 0\end{pmatrix}.$$

After simple calculations in the latter equality, we obtain that relation (23) holds if and only if the following equalities are true:

$$\begin{aligned}&\beta a(b\,{\rm e}^{{\rm i}\omega} - \overline{b}\,{\rm e}^{-{\rm i}\omega}) = 0, \qquad \beta d(b\,{\rm e}^{{\rm i}\omega} - \overline{b}\,{\rm e}^{-{\rm i}\omega}) = 0,\\ &4\alpha b\,{\rm e}^{{\rm i}\omega} = \beta((b\,{\rm e}^{{\rm i}\omega})^2 + ad) + 4\beta.\end{aligned} \tag{25}$$

Condition $b \neq 0$ and the third equality in (25) imply that $\beta \neq 0$. Hence, the first two relations in (25) are true if $a = d = 0$ or if ${\rm e}^{{\rm i}\omega} = |b|/b$. In the first case, (25) takes the form

$$4(\theta^2 + 1)b\,{\rm e}^{{\rm i}\omega} = (\theta^2 - 1)((b\,{\rm e}^{{\rm i}\omega})^2 + 4)$$

or $\theta^2(b\,{\rm e}^{{\rm i}\omega} - 2)^2 = (b\,{\rm e}^{{\rm i}\omega} + 2)^2$. It is easy to see that the latter relation has a positive solution θ^2 if and only if $b\,{\rm e}^{{\rm i}\omega}$ is real. So, we show that the parameter ω is determined by the relation ${\rm e}^{{\rm i}\omega} = |b|/b$ in any case where $b \neq 0$. But then the left-hand sides of the first two equalities in (25) vanish and the third equality can be rewritten as $\theta^2((|b| - 2)^2 + ad) = (|b| + 2)^2 + ad$. Obviously, the parameter θ^2 can be positive if and only if $D = ((|b| + 2)^2 + ad)((|b| - 2)^2 + ad) > 0$. In this case, $\theta = \sqrt{\frac{(|b|+2)^2+ad}{(|b|-2)^2+ad}}$ and ω is uniquely defined by the relation ${\rm e}^{{\rm i}\omega} = |b|/b$.

Thus, we establish that (23) has a unique solution $F_{\theta,\omega}$ (in the class of operators of the form (21)), where parameters θ and ω are determined by (22). Hence, A is similar to the Hermitian operator $H = \sqrt{F_{\theta,\omega}}A\sqrt{F_{\theta,\omega}}^{-1}$.

Using (21), it is easy to verify that

$$F_{\theta,\omega}F_{\theta,\omega} = F_{\theta^2,\omega} \qquad \text{and} \qquad F_{\theta,\omega}F_{\theta,\omega_1} = I \quad (\text{if } |\omega - \omega_1| = \pi). \tag{26}$$

Thus $\sqrt{F_{\theta,\omega}} = F_{\sqrt{\theta},\omega}$ and $\sqrt{F_{\theta,\omega}}^{-1} = F_{\sqrt{\theta},\omega_1}$. Theorem 6 is proved. □

5. $\mathcal{P}$-Hermitian operators with $\mathcal{C}$-symmetries

Proposition 1. *Let A be a $\mathcal{P}$-Hermitian operator acting in $L_2(\mathbb{R})$. Then the following statements are equivalent:*

1. A has the property of $\mathcal{C}$-symmetry.
2. A is similar to a Hermitian operator.

Proof. Let A have the property of $\mathcal{C}$-symmetry. It follows from (4) and condition (c) of the definition of $\mathcal{C}$-symmetries that $(f, g)_{\mathcal{C}} = (Ff, g)$, where $F = \mathcal{PC}$ is a bounded uniformly positive Hermitian operator. Furthermore, by virtue of condition (a), relation $\mathcal{P}^2 = I$ and (3) we establish that $A^*F = FA$. In view of lemma 1, this means that A is similar to the Hermitian operator $H = \sqrt{\mathcal{PC}}A\sqrt{\mathcal{PC}}^{-1}$. So, the implication 1 ⇒ 2 is proved.

Let a $\mathcal{P}$-Hermitian operator A be similar to a Hermitian operator. In this case (see [16]), there exist subspaces $\mathfrak{L}_\pm$ of $L_2(\mathbb{R})$ mutually orthogonal with respect to the indefinite metric (4) that are invariant with respect to A and such that: $\mathfrak{L}_+$ is positive with respect to (4) (i.e., $[f, f] > 0$ for all $f \neq 0$ from $\mathfrak{L}_+$), $\mathfrak{L}_-$ is negative (i.e., $[f, f] < 0$ for all $f \neq 0$ from $\mathfrak{L}_-$), and

$$L_2(\mathbb{R}) = \mathfrak{L}_- \dot{+} \mathfrak{L}_+. \tag{27}$$

Moreover, the projectors $P_{\mathfrak{L}_\pm}$ onto $\mathfrak{L}_\pm$ with respect to decomposition (27) have the form

$$P_{\mathfrak{L}_-} = \tfrac{1}{2}[I - \mathcal{P}(I-K)(I+K)^{-1}], \qquad P_{\mathfrak{L}_+} = \tfrac{1}{2}[I + \mathcal{P}(I-K)(I+K)^{-1}],$$

where K is an operator of transition in $L_2(\mathbb{R})$.

Let us verify that the operator

$$\mathcal{C} = P_{\mathfrak{L}_+} - P_{\mathfrak{L}_-} = \mathcal{P}(I-K)(I+K)^{-1} \tag{28}$$

is a $\mathcal{C}$-symmetry for A. Indeed, since $P_{\mathfrak{L}_\pm}$ are projectors on $\mathfrak{L}_\pm$, the equality $\mathcal{C}^2 = I$ is obvious and the relation $\mathcal{C}A = A\mathcal{C}$ follows from the invariance of A with respect to $\mathfrak{L}_\pm$. Moreover, using decomposition (27) and taking into account well-known results of the Krein spaces theory [17], it is easy to see that the sesqulinear form $(\cdot,\cdot)_{\mathcal{C}} = [\mathcal{C}\cdot,\cdot]$ determines an inner product in $L_2(\mathbb{R})$, which is equivalent to $(\cdot,\cdot)$. So, we show that (28) determines a $\mathcal{C}$-symmetry for A. Proposition 1 is proved. □

It follows from theorem 6 and proposition 1 that any $\mathcal{P}$-Hermitian operator A defined by (8) where parameters a, b, d satisfy the conditions $D > 0$, $b \neq 0$, possesses the property of $\mathcal{C}_{\theta,\omega}$-symmetry, where $\mathcal{C}_{\theta,\omega} = \mathcal{P}F_{\theta,\omega}$ and parameters θ, ω are defined by (22).

Note that the 2-parameter set $\{\mathcal{C}_{\theta,\omega}\}_{\omega\in[0,2\pi),\theta>0}$ determines sufficiently representative collection of $\mathcal{C}$-symmetries. Namely, the following theorem was proved in [18]. For the convenience of the reader some principal stages of the proof are recalled.

Theorem 7 ([18])**.** *If a $\mathcal{P}$-Hermitian operator realization A of (1) defined by (8) possesses the property of $\mathcal{C}$-symmetry, where $\mathcal{C}$ commutes with A_{sym}, then A also possesses the property of $\mathcal{C}_{\theta,\omega}$-symmetry for a certain choice of parameters $\theta > 0$ and $\omega \in [0, 2\pi)$.*

Proof. Let A have the property of $\mathcal{C}$-symmetry and $A_{\mathrm{sym}}\mathcal{C} = \mathcal{C}A_{\mathrm{sym}}$. In this case, (5) implies that $A_{\mathrm{sym}}F = FA_{\mathrm{sym}}$, where $F = \mathcal{P}\mathcal{C}$. Hence, the subspaces $\mathcal{H}_\gamma = \ker(A^*_{\mathrm{sym}} + \gamma I)$ $(\gamma > 0)$ reduce F. By conditions (b), (c) of the definition of $\mathcal{C}$-symmetries, F is a bounded uniformly positive Hermitian operator and $(\mathcal{P}F)^2 = I$. Hence, the restriction $F\upharpoonright_{\mathcal{H}_\gamma}$ is a bounded Hermitian operator in $\mathcal{H}_\gamma$ such that

$$F\upharpoonright_{\mathcal{H}_\gamma} > 0 \qquad \text{and} \qquad (\mathcal{P}F\upharpoonright_{\mathcal{H}_\gamma})^2 = I\upharpoonright_{\mathcal{H}_\gamma}. \tag{29}$$

In view of proposition 1, A is similar to a Hermitian operator. For this reason, A has a real spectrum and there exists $\gamma > 0$ such that $-\gamma \in \rho(A)$ (since A is a finite dimensional perturbation of the positive operator A_0). Let us fix such γ and consider the matrix representation $\mathbf{F}_\gamma = (F_{ij})_{i,j=1}^2$ of $F\upharpoonright_{\mathcal{H}_\gamma}$ with respect to the orthonormal basis

$$h_{1\gamma}(x) = \frac{1}{\gamma^{1/4}}\begin{cases} \mathrm{e}^{-\sqrt{\gamma}x}, & x > 0 \\ \mathrm{e}^{\sqrt{\gamma}x}, & x < 0 \end{cases} \qquad h_{2\gamma}(x) = \frac{1}{\gamma^{1/4}}\begin{cases} -\mathrm{e}^{-\sqrt{\gamma}x}, & x > 0 \\ \mathrm{e}^{\sqrt{\gamma}x}, & x < 0. \end{cases}$$

Since the matrix representation of $\mathcal{P}\upharpoonright_{\mathcal{H}_\gamma}$ with respect to this basis coincides with $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, we can reformulate conditions (29) as follows

$$F_{ii} > 0, \qquad F_{11}F_{22} - |F_{12}|^2 > 0, \qquad \begin{pmatrix} F_{11}^2 - |F_{12}|^2 & F_{12}(F_{11} - F_{22}) \\ -\overline{F_{12}}(F_{11} - F_{22}) & F_{22}^2 - |F_{12}|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Elementary analysis shows that these relations hold if and only if $\mathbf{F}_\gamma$ admits the representation

$$\mathbf{F}_\gamma = \begin{pmatrix} \mathfrak{f} & \mathfrak{g}\,\mathrm{e}^{-\mathrm{i}\varpi} \\ \mathfrak{g}\,\mathrm{e}^{\mathrm{i}\varpi} & \mathfrak{f} \end{pmatrix}, \qquad \mathfrak{f}^2 - \mathfrak{g}^2 = 1, \qquad \mathfrak{f} > 0, \qquad \mathfrak{g} \geqslant 0, \qquad \varpi \in [0, 2\pi). \tag{30}$$

On the other hand, relations (5) and (21) imply that the operators $F_{\theta,\omega}$ also commute with A_{sym} and the matrix representation $\mathbf{F}_{\theta,\omega}$ of $F_{\theta,\omega} \upharpoonright_{\mathcal{H}_\gamma}$ with respect to $\{h_{j\gamma}(x)\}_{j=1}^2$ has the form

$$\mathbf{F}_{\theta,\omega} = \frac{1}{2}\begin{pmatrix} \theta + 1/\theta & -(\theta - 1/\theta)\,\mathrm{e}^{-\mathrm{i}\omega} \\ -(\theta - 1/\theta)\,\mathrm{e}^{\mathrm{i}\omega} & \theta + 1/\theta \end{pmatrix}, \qquad \theta > 0, \qquad \omega \in [0, 2\pi). \tag{31}$$

Comparing (30) and (31), we obtain that $\mathbf{F}_\gamma$ coincides with $\mathbf{F}_{\theta,\omega}$ if we put $\theta = \mathfrak{f} - \mathfrak{g}$ and $\omega = \varpi$. This conclusion is a key point of the proof which enables one to use the results of [18] in order to complete the proof of theorem 7. Namely, it follows from [18, theorem 9] that the existence of a $\mathcal{P}$-Hermitian extension A of A_{sym} with $\mathcal{C}$-symmetry, where $F = \mathcal{PC}$ commutes with A_{sym} *is equivalent* to the existence of a Hermitian operator $M \upharpoonright_{\mathcal{H}_\gamma}$ acting in $\mathcal{H}_\gamma$ and such that $M\mathcal{P}F \upharpoonright_{\mathcal{H}_\gamma} = F\mathcal{P}M \upharpoonright_{\mathcal{H}_\gamma}$ or (passing to the matrix representation) to the existence of a Hermitian matrix $\mathbf{M} = (m_{ij})_{i,j=1}^2$ such that

$$\mathbf{M}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{F}_\gamma = \mathbf{F}_\gamma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{M}.$$

Replacing in the latter equality $\mathbf{F}_\gamma$ by $\mathbf{F}_{\theta,\omega}$ and reasoning in the inverse order, we arrive at the conclusion that the operator A also possesses the property of $\mathcal{C}_{\theta,\omega}$-symmetry, where $\mathcal{C}_{\theta,\omega} = \mathcal{P}F_{\theta,\omega}$. Theorem 7 is proved. □

Corollary 1. *If a $\mathcal{P}$-Hermitian realization A of (1) defined by (8) has a $\mathcal{C}$-symmetry, where $\mathcal{C}$ commutes with A_{sym}, then A is similar to a Hermitian extension of A_{sym}.*

Proof. By theorem 7, the operator A also possesses the property of $\mathcal{C}_{\theta,\omega}$-symmetry. In this case, using proposition 1 and relations (26), we arrive at the conclusion that A is similar to the Hermitian operator $H = F_{\sqrt{\theta},\omega} A F_{\sqrt{\theta},\omega_1}$. Since $F_{\theta,\omega}$ commutes with A_{sym} for any choice of $\theta > 0$, $\omega \in [0, 2\pi)$ and $F_{\sqrt{\theta},\omega} F_{\sqrt{\theta},\omega_1} = I$, the operator H is a Hermitian extension of A_{sym}. Corollary 1 is proved. □

In conclusion, we remark that, by proposition 1, any $\mathcal{P}$-Hermitian operator A defined by the conditions of theorem 5 also has a $\mathcal{C}$-symmetry. However such a symmetry *cannot commute* with A_{sym}.

Indeed, if we suppose the commutation of $\mathcal{C}$ and A_{sym}, then theorem 7 implies that A possesses the property of $\mathcal{C}_{\theta,\omega}$-symmetry and, hence, equality (23) holds for a certain $\theta > 0$ and ω. Starting from this equality and repeating the arguments of the proof of theorem 6, we arrive at the conclusion that the corresponding parameters a, b, d of A in (8) satisfy the condition $D > 0$, which contradicts conditions of theorem 5.

The question of explicit construction of $\mathcal{C}$-symmetries for $\mathcal{P}$-Hermitian realizations of (1) considered in theorem 5 is open.

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